6. Revisit of Derivative and the Proof of the Fundamental Theorem

We have defined the derivative $f'(x_0)$ as the slope of the function at x_0, which can be estimated by the ratio $\frac{f(x)-f(x_0)}{x-x_0}$ as shown before. **Notice that the slope of a tangent line is a geometric concept and we can not depend on it to calculate derivatives of sophisticated function effectively.** The limit is a perfect tool for us to describe the slope of tangent line, i.e. the derivative. We define the derivative as the following limit if it exists.

$$f'(x_0) := \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{6.1}$$

Let $\Delta x = x - x_0$, we can write above definition as

$$f'(x_0) := \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{6.2}$$

Remark 12. *It is not hard to show that $f(x)$ is continuous at x_0 if $f'(x_0)$ exists. The other way is not true with a counter example $f(x) = |x|$.*

If the derivative $f'(x_0)$ of a $f(x)$ exists for each point x_0 in $[a, b]$, we get a derivative function defined on $[a, b]$. Following the convention to use x as the variable of a general function,

Definition 6.1. *The* **derivative function** *$f'(x)$ is defined by*

$$f'(x) = \lim_{\Delta \to 0} \frac{f((x + \Delta)) - f(x)}{\Delta}, \quad x \in (a, b) \tag{6.3}$$

and

$$f'(a) = \lim_{\Delta \to 0^+} \frac{f((a + \Delta)) - f(a)}{\Delta}, \quad f'(b) = \lim_{\Delta \to 0^-} \frac{f((b + \Delta)) - f(b)}{\Delta},$$

We have the following basic properties of derivatives:

Theorem 6.2. *Assume that $f(x)$ and $g(x)$ are two functions and both $f'(x)$ and $g'(x)$ exist at x. then*

$$(cf)'(x) = cf'(x) \quad \text{for any constant } c \tag{6.4}$$

$$(f + g)'(x) = f'(x) + g'(x) \tag{6.5}$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \tag{6.6}$$

$$(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \quad \text{if } g(x) \neq 0 \tag{6.7}$$

where the operations of functions f and g are defined in a natural way as follows

$$\begin{aligned} (cf)(x) &:= cf(x), \quad (f + g)(x) := f(x) + g(x), \\ (fg)(x) &:= f(x)g(x), \quad (\frac{f}{g})(x) = \frac{f(x)}{g(x)}. \end{aligned}$$

Proof. All results are based on the properties of limits outlined in Theorem 5.3. Eq (6.4) is straight forward from (5.4). By the definition, if $f'(x_0)$ and $g'(x_0)$ exist, then

$$\begin{aligned}(f+g)'(x) &= \lim_{\Delta\to 0}\frac{(f+g)(x+\Delta)-(f+g)(x)}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{f(x+\Delta)+g(x+\Delta)-(f(x)+g(x))}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{f(x+\Delta)-f(x)}{\Delta}+\lim_{\Delta\to 0}\frac{g(x+\Delta)-g(x)}{\Delta}\\ &= f'(x)+g'(x)\end{aligned}$$

For Eq (6.6), we have

$$\begin{aligned}(fg)'(x) &= \lim_{\Delta\to 0}\frac{(fg)(x+\Delta)-(fg)(x)}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{f(x+\Delta)g(x+\Delta)-(f(x)g(x))}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{f(x+\Delta)(g(x+\Delta)-g(x))+g(x)(f(x+\Delta)-f(x))}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{f(x+\Delta)(g(x+\Delta)-g(x))}{\Delta}+\lim_{\Delta\to 0}\frac{g(x)(f(x+\Delta)-f(x))}{\Delta}\\ &= f(x)g'(x)+f'(x)g(x)\end{aligned}$$

For Eq (6.7),

$$\begin{aligned}(\frac{f}{g})'(x) &= \lim_{\Delta\to 0}\frac{(\frac{f}{g})(x+\Delta)-(\frac{f}{g})(x)}{\Delta}=\lim_{\Delta\to 0}\frac{\frac{f(x+\Delta)}{g(x+\Delta)}-\frac{f(x)}{g(x)}}{\Delta}\\ &= \lim_{\Delta\to 0}\frac{1}{g(x)g(x+\Delta)}\lim_{\Delta\to 0}\frac{f(x+\Delta)g(x)-f(x)g(x+\Delta)}{\Delta}\\ &= \frac{1}{g^2(x)}\lim_{\Delta\to 0}(g(x)\frac{f(x+\Delta)-f(x)}{\Delta}-f(x)\frac{g(x+\Delta)-g(x)}{\Delta})\\ &= \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}\end{aligned}$$

□

S: **Geometrically, it is not straightforward to me that the slope of a tangent line of $f(x)+g(x)$ can be calculated by (6.5), not even mention for much more complicated case $f(x)g(x)$ and $f(x)/g(x)$ in (6.6) and (6.7). Even now given with the answer, I wonder if a geometric proof is possible. Yet, using limit to describe the slope, everything becomes quite manageable if not trivial, and the argument applies to a general function!**

T: Excellent point! Geometrical view such as tangent line often can give us good intuitive idea, analyses based on algebraic operations, on the other hand, can help us to deepen our understanding and lay out a solid foundation. Again, we can depend on algebraic operations on limit and derivative simply because we have a clear answer to the very basic question: what exactly are they?

We are now very close to prove the fundamental theorem. To that aim, we need build connection between derivative and integration, the most two important quantities in calculus.

Theorem 6.3. *Let $f(x)$ be a continuous function over the range $[a, b]$, define the function $H(x)$ defined over $[a, b]$*

$$H(x) = \int_a^x f(t)dt \tag{6.8}$$

then

$$H'(x) = f(x), \quad \forall x \in [a, b] \tag{6.9}$$

Remark 13. *Notice that $H(x)$ represents the area of upper bounded by f over the range $[a, x]$. The area changes as x moves in the range $[a, b]$. As such, it defines a function of x over $[a, b]$.*

Proof. we need to prove that Eq (6.9) holds at any fixed $x_0 \in [a, b]$, i.e.

$$\lim_{x \to x_0} \frac{H(x) - H(x_0)}{x - x_0} = f(x_0). \tag{6.10}$$

By definition, For any given $\epsilon > 0$, we need show that there exist $\delta > 0$ such that

$$|\frac{H(x) - H(x_0)}{x - x_0} - f(x_0)| < \epsilon, \quad \forall x \in (x_0 - \delta, x_0 + \delta). \tag{6.11}$$

To that aim, let us first analyze the left side of Equation 6.11. We have

$$\begin{aligned}
\frac{H(x) - H(x_0)}{x - x_0} &= \frac{\int_a^x f(t)dt - \int_a^{x_0} f(t)dt}{\Delta x} \\
&= \frac{\int_{x_0}^x f(t)dt}{x - x_0} \qquad \text{by Eq (3.17)} \\
&= \frac{f(c)(x - x_0)}{x - x_0}, \quad \text{for some } c \text{ between } x_0 \text{ and } x \text{ by Eq (4.20)} \\
&= f(c)
\end{aligned} \tag{6.12}$$

Inequity 6.11 is reduced to

$$|f(c) - f(x_0)| < \epsilon, \quad \forall x \in (x_0 - \delta, x_0 + \delta). \tag{6.13}$$

S: But how can we deal with c in the left side? we only know it exists somewhere.

T: Good point. The goal is to find δ such that Inequity 6.11 holds for all x in $O(x_0, \delta)$. Although we have no idea about exact value of c, but we know that it is between x_0 and x [30]. Given above ϵ, since $f(x)$ is continuous, there exists δ_0 such that

$$|f(x) - f(x_0)| < \epsilon, \quad \forall x \in (x_0 - \delta_0, x_0 + \delta_0). \tag{6.14}$$

S: Wait. if x is in $O(x_0, \delta_0)$, the corresponding c must be in $O(x_0, \delta_0)$ as well since it is more loser to x_0 than x, which implies $|f(c) - f(x_0)| < \epsilon$ by 6.14. should we just pick $\delta = \delta_0$?

T: What we just did is a typical analysis. Know what you want and find the answer by playing around with such arguments as "A is equivalent to B", or "A is true if we can prove B". Know how to select δ, let us finish the prove with a formal argument as follows.

For any given $\epsilon > 0$, since $f(x)$ is continuoue at x_0, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon, \quad \forall x \in (x_0 - \delta, x_0 + \delta). \tag{6.15}$$

Therefore, we have

$$\begin{aligned} \left|\frac{H(x) - H(x_0)}{x - x_0} - f(x_0)\right| &= |f(c) - f(x_0)| \quad \text{by Eq. 6.12} \\ &< \epsilon \qquad \text{since } |c - x_0| \le |x - x_0| < \delta \text{ and Eq. 6.15}, \end{aligned}$$

and therefore Eq. 6.10 holds by definition. □

The proof of the fundamental theorem 3.3. The fundamental theorem becomes trivial as a corollary of Theorem 6.3. Let $F(x)$ be any function such that $F'(x) = f(x)$. Define

$$G(x) = F(x) - \int_a^x f(t)dt$$

then $G'(x) = f(x) - f(x) \equiv 0$ and therefore $G(x)$ is constant by Theorem 2.3, Part 4. Notice that $G(a) = F(a) - 0 = F(a)$, therefore $G(x) \equiv F(a)$ and

$$\int_a^b f(t)dt = F(b) - G(b) = F(b) - F(a)$$

which implies Eq (3.18).

S: Wonderful! I cannot believe I could go through all the details and reach to this milestone.

T: **You do not see many equations as beautiful like Eq (3.18). The right hand is as complicated as the limit of a complex summation and the left side is as simple as the difference of two function values.**

S: You have fully convinced me with your opening statement now. So what is next magic?

T: We are going to decompose a general function as the summation of simple functions as in Eq 1.13. Just be a little patient and we are very close to reach a milestone!

[30] or between x and x_0 if $x < x_0$