5. Limit of Functions

T: We have clarified what it exactly means for a function $f(x)$ to be continuous at a point x_0 and understand what exactly means of the sentence: $f(x)$ approaches arbitrarily close to $f(x_0)$ if x is sufficiently close to x_0. In some cases, a function $f(x)$ is not well-defined at x_0 and yet we are interested in the limit of $f(x)$ as x approaches to x_0 without touching x_0. In Discussion Question 2.1, we ran into this situation where we estimate the slope of tangent line at $(x_0, f(x_0))$ by $\frac{f(x)-f(x_0)}{x-x_0}$ for $f(x) = x^2$, which can approach arbitrarily close to $2x_0$ as x goes to x_0 but not touching x_0. By some sandwich-type analysis, we showed that $2x_0$ must be the desired slope.

We need to generalize the concept of continuity to allow that $f(x)$ might not be defined at x_0. Let $\hat{O}(x_0, r) := (x_0 - r, x_0 + r) - x_0$ denote a neighborhood of x_0 without x_0.

Definition 5.1. *$f(x)$ is called to converge to a value l as x approaches x_0 if for any prescribed $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|f(x) - l| < \epsilon, \quad \forall x \in \hat{O}(x_0, \delta) \tag{5.1}$$

and we write $\lim_{x\to x_0} f(x) = l$

Remark 10.
1. *Compared to Definition 4.1 for $f(x)$ to be continuous at x_0, there are two differences for $f(x)$ take limit l at x_0 : 1) the limit value l is not required to be the function value $f(x_0)$; 2) there is no requirement for how $f(x_0)$ is close to l. In fact, $f(x_0)$ might not even be defined.*
2. *It is clear that $f(x)$ is continuous at x_0 if and only if*

$$\lim_{x\to x_0} f(x) = f(x_0)$$

In some cases, we consider the limit value of $f(x)$ as x approaches x_0 only from the left or the right.

Definition 5.2.
1. *$f(x)$ is called to converge to a value l as x approaches x_0 from left if for any given ϵ, there exists $\delta > 0$ such that*

$$|f(x) - l| < \epsilon, \quad \forall x \in (x_0 - \delta, x_0) \tag{5.2}$$

and we write $\lim_{x\to x_0^-} f(x) = l$.

2. *$f(x)$ is called to converge to a value l as x approaches x_0 from right if for any prescribed ϵ, there exists $\delta > 0$ such that*

$$|f(x) - l| < \epsilon, \quad \forall x \in (x_0, x_0 + \delta) \tag{5.3}$$

and we write $\lim_{x\to x_0^+} f(x) = l$

3. *If $\lim_{x\to x_0^-} f(x) = f(x_0)$, we say that $f(x)$ is left continuous at $x = x_0$. Similarly, we say that $f(x)$ is right continuous at $x = x_0$ if $\lim_{x\to x_0^+} f(x) = f(x_0)$.*

Remark 11. 1. *It is clear that* $\lim_{x\to x_0} f(x) = l$ *if and only if*

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = l.$$

2. *If both* $\lim_{x\to x_0^-} f(x)$ *and* $\lim_{x\to x_0^+} f(x)$ *exist, but they are not equal. we say that* $f(x)$ *has a jump at* x_0*. Take the function* $f(x)$ *shown in Figure 16, it is clear that there is jump at* $x = 1$ *and* $f(x)$ *is right continuous at* $x = 1$*.*

The following properties about the limit should not be surprising since the underlying idea of limit basically implies "$f(x) \to l$ as $x \to x_0$".

Theorem 5.3. *Assume that* $\lim_{x\to x_0} f(x) = l$ *and* $\lim_{x\to x_0} g(x) = m$ *both exist, then*

$$\lim_{x\to x_0} cf(x) = c \lim_{x\to x_0} f(x) = cl \quad \textit{for any constant } c \tag{5.4}$$

$$\lim_{x\to x_0} (f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x) = l + m \tag{5.5}$$

$$\lim_{x\to x_0} (f(x)g(x)) = \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x) = lm \tag{5.6}$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{l}{m} \quad \textit{if } m \neq 0 \tag{5.7}$$

One can show Eq (5.5) by the definition in same way as we did in Proposition 4.2, i.e. showing the existence of a desired δ for any prescribed $\epsilon > 0$. The other properties in above theorem can also be approved in the same $\epsilon - \delta$ fashion. The details are left as exercises.

S: It might take me sometime to figure out proves. Those equations make sense to me. Should we just accept them?

T: These are part of the foundation. Playing with the proves are actually good exercise for you to get familiar with limit, the very key concept in calculus, and become comfortable with the $\delta - \epsilon$ language that we often use in the development of calculus.

The results in Theory 5.3 are important since it makes possible for us to handle the limit of more sophisticated functions that are constructed through algebraic operations of simple functions.

S: I see. So we can handle limits not only of polynomials, but of ratio of polynomials.

T: Correct. We shall extend the building blocks of functions and include exponential functions, logarithm functions, and trigonometric functions. You will see that we can effectively handle quite wide spectrum of functions with the properties of the limit described in Theorem 5.3.