4. Continuous functions

T: In general, function can be as bizarre as the Dirichlet function in Example 3. In this section, we shall restrict our discussion to a certain regular function, so called continuous functions. Continuity is a geometric concept when we view a function $f(x)$ as a curve $(x, y = f(x))$ in the xy plan. What can you say about the continuity of $y = f(x)$ as described in Figure 16?

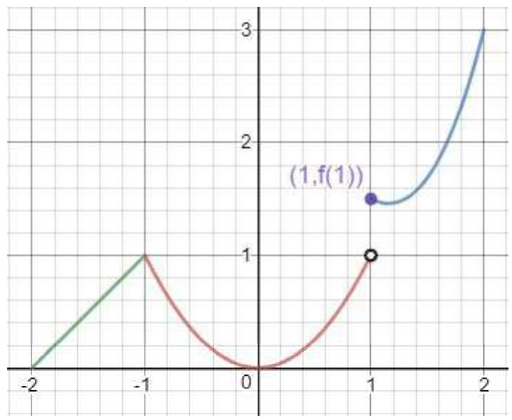

FIG 16. *$y = f(x)$ is defined $[-2, 2]$ and is continuous everywhere except at $x = 1$*

S: I am not sure how to describe the continuity in a precisely way. But I can visualize that it is continuous everywhere except at $x = 1$, where there is a jump.

T: A good starting observation. At $x = 1$, a small change, denoted by $\Delta x := x - 1$, in x coordinate might lead to significant change, denoted by $\Delta y := f(1 + \Delta x) - f(1)$, in y coordinate on the curve.

S: OK. So we can say that at any other point $x_0 \neq 1$ in the domain $D_f = [-2, 2]$, $y = f(x)$ is continuous because a small change $\Delta x = x - x_0$ leads to a small change $\Delta y := f(x_0 + \Delta x) - f(x_0)$.

T: But how to define a change is **small**.

S: Ooh, I should use the word "arbitrary close". How about $f(x)$ approaches arbitrarily close to $f(x_0)$ as x is sufficiently close to x_0.

T: That is the correct language in English. Let me use math language to clarify the continuity of $f(x)$ at x_0 and then explain it. Notice that it is one of the most important concepts in calculus, if not THE one.

Definition 4.1. *A function $f(x)$ is called to be continuous at x_0 if for* **any** *given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$|f(x) - f(x_0)| < \epsilon, \quad \textit{for } \textbf{all } x \in (x_0 - \delta, x_0 + \delta). \tag{4.1}$$

If $f(x)$ is continuous at x_0, we write

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Remark 9. *1. Recall that for sequence $A = \{a_n\}_{n\geq 1}$ to converge to some value l, a_n is requested to approaches to l as n increases. Similarly, for a function f to be continuous at a given point x_0 in its domain, $f(x)$ need approach to $f(x_0)$ as x approaches to x_0.*

2. *Similar to the limit of a sequence, there is no restriction on the selection of $\epsilon > 0$ to reflect the requirement that $f(x)$ can be* **arbitrarily** *close to $f(x_0)$.*
3. *For a (arbitrarily) given ϵ, the continuity of f at x_0 demands the existence of δ with Condition (4.1). Notice we do not care much how close $f(x)$ is to $f(x_0)$ if x is outside of the interval $(x_0 - \delta, x_0 + \delta)$. i.e. $f(x)$ is required to be ϵ (arbitrarily) close to $f(x_0)$ if x is δ (sufficient) close to x_0.*
4. *For a given ϵ, the condition (4.1) need to be satisfied for* **all** *points in an identified δ neighborhood. If one neighborhood $O(x_0, \delta)$ meets the condition (4.1), any smaller small neighborhood like $O(x_0, \delta/2)$ certainly works as well. The value of δ is not important. What matters is the existence of such δ.*

Let us apply it to our simple function $y = x^2$.

Example 6. *Discuss the continuity of $f(x) = x^2$ at $x = 2$ by definition.*

We know by intuition $\lim_{x\to 2} x^2 = f(2) = 4$. To show it by definition, for **any** *$\epsilon > 0$, we need to find δ such that*

$$|x^2 - 2^2| < \epsilon, \quad \forall x \in (2 - \delta, 2 + \delta). \tag{4.2}$$

For the **given** *ϵ, we can see that $|x^2 - 2^2| = |(x - 2)(x + 2)|$ can be smaller than ϵ if $|x + 2|$ is bounded and $|(x - 2)|$ sufficient small. As the first step to estimate $|x^2 - 2^2|$, we need an upper bound of $|x + 2|$. Since we can freely select δ, we first require*

$$\delta < 1, \tag{4.3}$$

which implies $1 < x < 3$ and therefore $|x + 2| < 5$. Under the assumption (4.3), $|x^2 - 2^2| < 5|x - 1|$. Now Inequity (4.2) holds for any δ such that

$$\delta < \frac{1}{5}\epsilon. \tag{4.4}$$

It is clear now any δ that meets the conditions (4.3) and (4.4) should work for our purpose. One can simply pick

$$\delta = \min\{1, \frac{1}{5}\epsilon\}. \tag{4.5}$$

i.e. delta is chosen to be the minimum of 1 and $\frac{1}{5}\epsilon$. To verify, we have

$$\begin{aligned} |(x^2 - 2^2)| &= |(x - 2)(x + 2)| \\ &< 5|(x - 2)|, \quad \textit{since } |x + 2| < 5 \textit{ due to } |x - 1| < \delta \leq 1 \\ &< 5 \times \delta \leq \epsilon, \quad \textit{since } |x - 2| < \delta \leq \tfrac{\epsilon}{5} \end{aligned}$$

S: It looks that we depend on some algebra to do some analysis to find a suitable δ.

T: Finding δ for a given ϵ need some algebraic skills since the value of ϵ can be any positive number. But we don't need start from definition to show the continuity in practice since there are effective rules for us to justify the continuity of given functions.

Let us prove one of those rules to get familiar with $\epsilon - \delta$ language.

Proposition 4.2. *Let $f(x)$ and $g(x)$ be both continuous at x_0, then $h(x) := f(x) + g(x)$ is also continuous.*

Proof. As always, we start with any given $\epsilon > 0$ and look for $\delta > 0$ such that

$$|h(x) - h(x_0)| < \epsilon, \qquad \forall x \in (x_0 - \delta, x_0 + \delta). \tag{4.6}$$

Since $f(x)$ is continuous at x_0, there exists δ_1 associated to the given positive number $\epsilon_0 := \frac{1}{2}\epsilon$ such that

$$|f(x) - f(x_0)| < \epsilon_0 = \frac{1}{2}\epsilon, \qquad \forall x \in (x_0 - \delta_1, x_0 + \delta_1). \tag{4.7}$$

Similar arguments applying to $g(x)$, there exists δ_2 associated to the given positive number $\epsilon_2 := \frac{1}{2}\epsilon$ such that

$$|g(x) - g(x_0)| < \epsilon_0 = \frac{1}{2}\epsilon, \qquad \forall x \in (x_0 - \delta_2, x_0 + \delta_2). \tag{4.8}$$

We show that

$$\delta = \min\{\delta_1, \delta_2\} \tag{4.9}$$

meet the requirement (4.6). In fact, for $x \in (x_0 - \delta, x_0 + \delta)$

$$\begin{aligned} |(h(x) - h(x_0))| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \epsilon_0 \quad \text{since } |x - x_0| < \delta < \delta_1 \text{ and } (4.7) \\ &+ \epsilon_0 \quad \text{since } |x - x_0| < \delta < \delta_2 \text{ and } (4.8) \\ &= \epsilon \end{aligned}$$

□

S: The way of finding δ is interesting. A pure logic proof for the existence of some desired quantity.

T: Right. For the given ϵ, the existence of δ is based on the existence δ_1 and δ_2 whose existence is due to the fact that f and g are continuous at x_0.

The clarification of the continuity in Definition 4.1 makes calculus build on a solid foundation so we can effectively handle general functions, a task that is hard to achieve by depending on intuitive geometric arguments.

Let us look another interesting example to get familiar with the concept. Remember that any ration number can be uniquely expressed as $\frac{p}{q}$ such that there is no common factor of p and q except 1.

Example 7. *Discuss the continuity of the following function defined over the range* $[0,1]$.

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ is a rational number over } [0,1] \text{ ;} \\ 0, & \text{if } x \text{ is a irrational number over } [0,1]. \end{cases} \tag{4.10}$$

Remind you that rational numbers and irrational numbers are dense in the real number set R, i.e. we can always find a rational number or an irrational number over any interval (a,b)*. If* $x_0 = \frac{p}{q}$ *is a rational number in* $[0,1]$*, we claim that there is no* δ *that meets for Condition 4.1 for* $\epsilon = \frac{1}{2q}$*. In fact, for any* δ*, we can always find an irrational number* $x \in (x_0-\delta, x_0+\delta)$*, but*

$$|f(x) - f(x_0)| = |0 - \frac{1}{q}| = \frac{1}{q} > \epsilon = \frac{1}{2q}.$$

As such, we conclude that $f(x)$ *is not continuous at any rational number. Now, for an irrational* x_0*, for any given* $\epsilon > 0$*, there are only finite rational numbers* $0 \le \frac{p}{q} \le 1$ *such that* $f(\frac{p}{q}) = \frac{1}{q} \ge \epsilon$*. we choose* δ *such that* $(x_0-\delta, x_0+\delta)$ *does not contain any of those numbers and therefore*

$$|f(x) - f(x_0)| = \begin{cases} \frac{1}{q} < \epsilon, & \text{if } x \in (x_0-\delta, x_0+\delta) \text{ is rational number } \frac{p}{q}; \\ 0 < \epsilon, & \text{if } x \in (x_0-\delta, x_0+\delta) \text{ is a irrational number }. \end{cases}$$

implies that $f(x)$ *is continuous at any irrational number.*

S: The function is what you called bizarre function. The result is interesting, but do we really need such function in real application?

T: there are reasons why we study such a blizzard function. First of all, the example shows that **it does not need to be "look continuous" for a function to be continuous.** Secondly, even if we are mainly interested in regular functions, those bizarre ones provide some counterexamples to tell you the limit of your theory.

$f(x)$ in Figure 16 is so called right continuous at 1 although it is not continuous. Let us extend the continuity concept slightly as follows.

Definition 4.3. *A function* $f(x)$ *is called to be* **right continuous** *at* x_0 *if for any given* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$|f(x) - f(x_0)| < \epsilon, \quad \text{for all } x \in [x_0, x_0+\delta) \tag{4.11}$$

If $f(x)$ *is right continuous at* x_0*, we write*

$$\lim_{x \to x_0^+} f(x) = f(x_0)$$

Similarly, $f(x)$ *is called to be* **left continuous** *at* x_0 *if for any given* $\epsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$|f(x) - f(x_0)| < \epsilon, \quad \text{for all } x \in (x_0-\delta, x_0] \tag{4.12}$$

If $f(x)$ *is left continuous at* x_0*, we write*

$$\lim_{x \to x_0^-} f(x) = f(x_0)$$

Assumption 4.1. *For the rest of the section, $f(x)$ is assumed to be continuous at any point $x \in (a,b)$, left continuous at b and right continuous at a. We simply say that $f(x)$ is continuous over $[a,b]$.*

For such function, we have

Theorem 4.4. *(Intermediate Value Theorem) If $f(x)$ is continuous over $[a,b]$, then for any value c between $f(a)$ and $f(b)$, there exists $x_0 \in [a,b]$ such that*

$$f(x_0) = c. \tag{4.13}$$

Proof. Let $g(x) := f(x) - c$. We need to show that there is a point x_0 such that $g(x_0) = 0$. To search such x_0, we split evenly the interval $[a,b]$ to two intervals and can take one of them, label as $[a_1, b_1]$, such that $g(a_1)g(b_1) \leq 0$ since $g(a)g(b) \leq 0$ [25]. Repeat the same process and we obtain a interval sequence such that for $n = 1, 2, ...$

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n], \tag{4.14}$$

$$b_n - a_n = \frac{1}{2^n}(b-a), \tag{4.15}$$

$$g(a_n)g(b_n) \leq 0. \tag{4.16}$$

By (4.14) and (4.15), there exists a unique x_0 that lies in all interval $[a_n, b_n]$ [26]. We claim that $f(x_0) = 0$. Otherwise, assuming $f(x_0) > 0$ [27], by the continuity of $f(x)$ at x_0, taking $\epsilon = \frac{f(x_0)}{2}$ [28], there is a $\delta > 0$ such that

$$-\frac{f(x_0)}{2} = -\epsilon < f(x) - f(x_0) < \epsilon = \frac{f(x_0)}{2}, \quad x \in (x_0 - \delta, x_0 + \delta),$$

which implies $f(x) > \frac{f(x_0)}{2} > 0$ for all $x \in O(x_0, \delta)$, i.e. $f(x)$ remains same sign in the neighborhood. Since $[a_n, b_n]$ shrinks to x_0, one can find $[a_n, b_n] \subset (x_0 - \delta, x_0 + \delta)$ and therefore $g(a_n)g(b_n) > 0$, which is contradictory to (4.16). As such, $f(x_0) = 0$. □

S: Interesting. I guess it is so called analysis you mentioned before. It starts searching the desired value x_0 by using a sequence of intervals $[a_n, b_n]$ that

[25] due to c is between $f(a)$ and $f(b)$

[26] The following property of the set R of the real numbers is accepted for grant as an axiom and the theory of calculus is built on it. If R is interpreted as a line, the axiom basically says that there is no hole in the line.

Axiom 4.1. *For a sequence of close intervals $\{[a_n, b_n]_{n\geq 1}\}$ such that*

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n], \quad \textit{and} \quad \lim_{n\to\infty} b_n - a_n = 0$$

then there exists a unique number c such that c lies in all intervals $[a_n, b_n]$.

Notice that the assumption of the close interval is necessary. It is easy to see that the intersection of all $(0, \frac{1}{n})$ for $n \geq 1$ is empty.

[27] Same argument can be applied to the case $f(x_0) < 0$.

[28] $\epsilon = -f(x_0)/2$ if we assume $f(x_0) < 0$

shrink to it; and then shows $f(x_0) = 0$ by contradiction: for if otherwise, $f(x)$ has to keep the same sign in a neighborhood of x_0 since $f(x)$ is continuous, which is contradictory to the fact $f(a_n)$ and $f(b_n)$ always have opposite signs and a_n and b_n fall in any given neighborhood of x_0 for sufficient large n.

T: Right. Analysis enables us to build a solid foundation of calculus. Let us apply similar analysis to prove some other important results for general continuous functions on $[a, b]$.

Theorem 4.5. *If $f(x)$ is continuous over $[a, b]$, then*

1. *$f(x)$ is bounded, i.e. there exists $M \geq 0$ such that $|f(x)| \leq M$ for any $x \in [a, b]$.*
2. *there exists $x_0 \in [a, b]$ such that $f(x_0) = \inf\{f(x), \quad x \in [a, b]\}$.*
3. *there exists $x_1 \in [a, b]$ such that $f(x_1) = \sup\{f(x), \quad x \in [a, b]\}$.*

where, for any set A, $\inf A$ and $\sup A$ are defined in Definition 1.1.

S: Can you review the differences between the min(max) and inf (sup) of a set A? somehow I am still not comfortable with $\sup A$ and $\inf A$.

T: OK. take $\sup A$ for a general set A, it is defined as the least upper bound of A [29] and is equal to $\max A$ if the maximum value of A exists. Notice that the maximum value is required to be in A by definition and exist always if A is a finite set. On the other hand, if A contains infinite many members, the maximum value of a set might not exist, which leads to the concept $\sup A$. Think about the set defined by all values of following function:

$$f(x) = \begin{cases} x, & x \in [0, \frac{1}{2}); \\ 0, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

S: I see. The set $f[0, 1]$ is $[0, \frac{1}{2})$, no member in it can be treated as the maximum value since $\frac{1}{2}$ is not included. $\frac{1}{2}$ is the *sup*, but not as the maximum value.

T: Correct. Theorem 4.5 actually says that the $\sup f[a, b]$ and $\inf f[a, b]$ are actually the maximum value and minimum value of f if it is continuous. The conclusion is not true in general as shown in about example, which is not continuous at $x = \frac{1}{2}$. Let us prove the theorem to get familiar with the analytic tricks we applied before.

Proof. 1. If M does not exists, then for any natural number n, n is not a upper bound and hence there is exist $x_n \in [a, b]$ such that $|f(x_n)| > n$. As such, we obtain a sequence $x_n \in [a, b]$ that meet the condition

$$|f(x_n)| > n, \quad n = 1, 2, \ldots. \tag{4.17}$$

We apply the same half-interval-split method in the proof of Theorem 4.4 to generate a sequence of intervals $([a_n, b_n])_{n \geq 1}$ such that each $[a_n, b_n]$

- contains infinite many of items in the sequence $\{x_n\}_{n \geq 1}$;

[29] if A is not upper bounded, $\sup A := \infty$ as defined in Definition 1.1

- both a_n and b_n go to x_0 due to $b_n - a_n \to 0$.

Since $f(x)$ is continuous at x_0, for $\epsilon = 1$, there exists a neighborhood $O(x_0, \delta)$ such that $|f(x) - f(x_0)| < 1$ or

$$|f(x)| < |f(x_0)| + 1 \qquad \text{for } x \text{ in } O(x_0, \delta). \tag{4.18}$$

i,e. $f(x)$ is bounded by $|f(x_0)| + 1$ over $O(x_0, \delta)$. Which is contradictory to the fact that there are infinite many points of x_n in $O(x_0, \delta)$ and $f(x)$ over those points are not bounded due to Condition (4.17).

2. Since $f(x)$ is bounded, $l := \inf\{f(x), \quad x \in [a, b]\}$ exists. For each $\frac{1}{n}$, since l is the greatest lower bound, we can find an x_n such that

$$l \leq f(x_n) \leq l + \frac{1}{n}. \tag{4.19}$$

Apply the same strategy in the proof of Part I, we can identify $x_0 \in [a, b]$ such that any neighborhood $O(x_0, \delta)$ of x_0 contains infinite many of members of the set $\{x_n\}_{n \geq 1}$. Note that $f(x_0) \geq l$ since l is a lower bound. We claim that $f(x_0) = l$. Otherwise $f(x_0) > l$, let $\epsilon_0 = \frac{f(x_0) - l}{2} > 0$, there exists a neighborhood $O(x_0, \delta_0)$ such that $f(x) - f(x_0) > -\epsilon_0$ for all $x \in O(x_0, \delta_0)$, or

$$f(x) > f(x_0) - \epsilon_0 = \frac{f(x_0) + l}{2} = l + \epsilon_0.$$

On the other hand, the set $\{x_n\}_{n \geq 1}$ contains infinite many elements and we can pick one x_n with large index n such that $\frac{1}{n} < \epsilon_0$. By above inequity,

$$f(x_n) > l + \epsilon_0 > l + \frac{1}{n}$$

which is contradictory to Eq 4.19.

3. Similar arguments in proving Part II can be applied to show Part III.

□

S: Seems that you play the same trick in above arguments: search a target point by constructing a sequence of intervals such that the point is contained in all of those intervals, the intervals eventually shrink to it.

T: Indeed. Once the point x_0 is identified, we can show that it is the desired point by the continuity of $f(x)$ at x_0. Combining Theorem 4.4 and Theorem 4.5, we have

Corollary 4.6. *Let $f(x)$ be continuous over $[a, b]$. Then*

1. *it reaches the maximum value M and the minimum value m, i.e. there exists $x_1, x_2 \in [a, b]$ such that*

$$f(x_1) = m, \quad f(x_2) = M;$$

2. *for any value $c \in [m, M]$, there exists $x_0 \in [a, b]$ such that $f(x_0) = c$.*

We apply above corollary to show another important mean theorem as demonstrated in Figure 17.

Theorem 4.7. *(Mean Value Theorem of integrals) If $f(x)$ is continuous over $[a, b]$, then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)dx = f(c)(b - a) \tag{4.20}$$

Proof. By Inequity 3.16, we know that

$$f(x_{min}) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f(x_{max})$$

and by Corollary 4.6, there exists $c \in [a, b]$ such that Equation (4.20) is satisfied. □

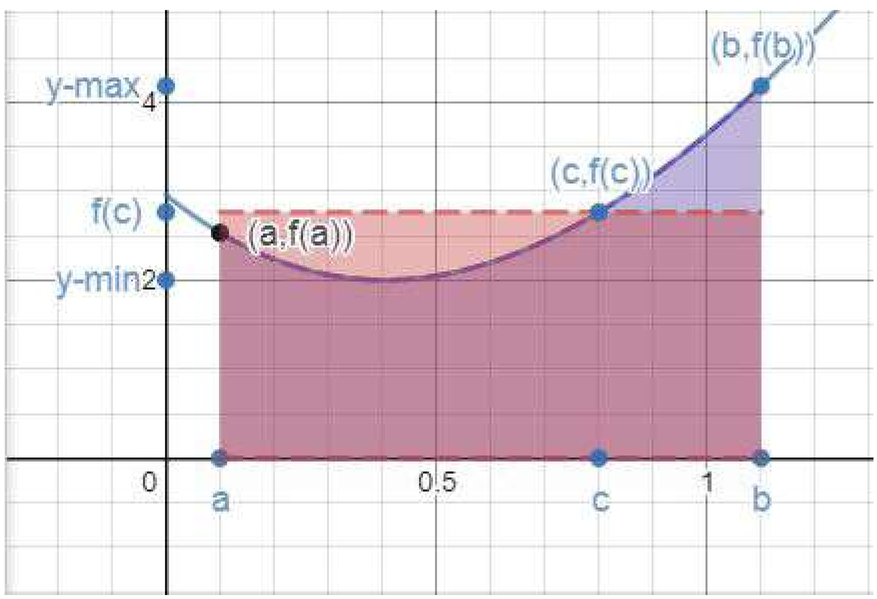

FIG 17. *Mean value theorem of integral for a continuous function $f(x)$ over $[a, b]$. There exists $c \in [a, b]$ such that the area $f(c)(b-a)$ of the rectangle is equal to the area of the shape bounded by $f(x)$, $x = a$, $x = b$ and x axis.*

S: I did not expect such simple proof for a general function. It looks that $f(c)$ represents the mean value of the function over $[a, b]$ as the name of the theorem suggest.

T: Looks like you start to appreciate the beauty part of the calculus. **It provides the logic for what looks right mainly because we are able to clarify what the continuity of a function is about.** As you point out, $f(c)$ can be interpreted as the mean value of a function over the range $[a, b]$. Can you image how to average function value over an interval without calculus as the tool?

S: Agree with you now that calculus is indeed a powerful tool to handle functions.