1. Introduction to limit – the core concept in the infinite world

Teacher (T): Well, Calculus might be the most important subject in math. As you might know, Newton is remembered as the most greatest scientist, but some people believe that his most important achievement is on the calculus. British people are quite proud for Newton's achievement. But folks in the European Continent are not quite happy with that and they claimed that German mathematician Leibniz should take that credit and argued for a long time with British people in history.

Student (S): I wish I were as enthusiastic like you on calculus. But can you briefly explain what it is about.

T: Generally speaking, it is hard to summarize in one sentence what a math subject is about, especially for beginners. Calculus is however special.

Statement 1.1. *Calculus is all about* **limit***, which bridges a* finite *world and an* infinite *world.*

S: You sound like a philosopher rather than a mathematician. Well, hope I would know my *limit* at the end. But how do you define the **infinite** world?

T: Let us walk in the finite world first, a green world that you should be comfortable to deal with now. Let $A_N = (a_1, a_2, ..., a_N)$ denote a list of finite many numbers. We can operate on the elements in A_N in some ways we are familiar with. For examples, we can sum them up:

$$\sum_{1\le n\le N} a_n := a_1 + a_2 + \cdots + a_N.$$

We can also single out the maximum value of A_N. It might take us some time, but we can always get an answer in the end. can't we?

S: Well, at least I can compare each number with all others to identify the maximum if I have enough time.

T: That is true. In the finite world, for a task like comparison, one can naively exhaust all possible scenarios to find the answer, a task computers can handle more efficiently than human beings. Now let us move to the **infinite** world, and consider a sequence of numbers that never stops, denoted by

$$A := (a_n)_{n\ge 1} = (a_1, a_2, ..., a_n, ...). \tag{1.1}$$

For example, A stands for the list of all natural numbers if $a_n = n$, arranged in increasing manner.

S: So what distinguishes the infinite world from the finite world has nothing to do with magnitude of certain quantity. In the infinite world, we need operate on infinite many of animals.

T: Be careful about the difference between two concepts: infinite many vs "infinity". The former should be clear in word itself, i.e. not finitely many. It is not easy to explain the latter exactly at this moment. Intuitively, think about what the following two sequences might represent?

$$(1, 2, \dots, n, \dots), \quad (1^2, 2^2, \dots, n^2, \dots). \tag{1.2}$$

Both sequences "approaches" to certain kind of infinity, denoted by ∞, although the second one "goes" to ∞ faster than the first one. Notice that concept of infinity does not refer to a *single* abstract quantity, but refer to certain kind of sequences that can go beyond any specific boundary. We shall clarify what it means exactly later. You make a good comment. In the infinite world, we often need to deal with infinite many animals or, more accurately, end up in an endless process, which distinguish from the finite world we are familiar with.

Let S to denote a general set of real numbers, including the case when its elements can not be listed in a sequence such as the open interval $S = (1, 2)$ [1]. Here comes the first basic concept in the infinite world.

Definition 1.1. *1. A number m is called as a* **lower bound** *of S if $m \leq x$ for ANY $x \in S$* [2]*. S is called* **lower bounded** *if such a lower bound exists;*
2. A value M is called as a **upper bound** *of S if $M \geq x$ for ANY $x \in S$. S is called* **upper bounded** *if such a upper bound exists;*
3. S is called **bounded** *if it is both lower bounded and upper bounded.*

Remark 1. *1. Above lower bound m or upper bound M of S is not required to be in S. For example, $S = (1, 2)$ is bounded with 0 and 3 as a lower bound and upper bound respectively. 0 and 3 are not in S.*
2. lower bound and upper bound are not unique. 1 and 2 are another lower bound and upper bound for above S.

Notice that a finite set is always bounded, the minimum value is a lower bound and the maximum value is a upper bound. As such, bound concept is only needed when we deal with a set S with infinite many elements. It is easy to see that both examples listed in Expression (1.2) are lower bounded, but not upper bounded.

S: Looks like "any" in definition 1.1 is the key word to understand lower bound and upper bound.

T: the logic of "any" need to be emphasized in several key concepts in calculus. Any idea if a number m is not a lower bound of S?

S: There should exist at least one number $x \in S$ such that $x < m$.

T: Great! let us write it down

Statement 1.2. *If m is not a lower bound of a set S, there exist $x \in S$ such that $x < m$. Similarly, if M is not a upper bound, there exist $y \in S$ such that $M < y$.*

As another example, let $a_n = (1/2)^n$ in (1.1), we get a geometric sequence

$$G(1/2) := (1, 1/2, (1/2)^2, \ldots, (1/2)^{n-1}, \ldots). \tag{1.3}$$

Notice that $G(1/2)$ is decreasing and hence the maximum value is $a_1 = 1$. what can we say about the minimum value of the set $G(1/2)$?

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[1] See Section 0.2 for the conventions and notations. Notice that $\{1, 2\}$ is used to denote the set of two numbers 1 and 2.

[2] The expression $x \in S$ refers to "x in S", see Section 0.2 for the conventions and notations

S: Well, $a_n = (1/2)^n$ gets close to 0 as n increases. So any positive number can't be treated as the minimum value, 0 and negative values are not in the sequence. Seems that there is no minimum value for $G(1/2)$, isn't it?

T: Excellent analysis!

S: 0 is a low bound since all elements in $G(1/2)$ are larger than 0. Of course, any negative number is a low bound as well.

T: What makes 0 special is that it is **the greatest lower bound** of $G(1/2)$.

S: I see. For any other *larger* number, it can not be treated as a lower bound of $G(1/2)$. is this the logic by saying "greatest lower bound"?

T: Exactly! we can transform the phrase to a mathematical description that is quite helpful in future analysis:

Definition 1.2. *Let S be a general set of numbers,*

1. *A number m is called the greatest lower bound of S if*
   - *$m \leq x$ for all x in S* [3]*;*
   - *For any positive number ϵ, $m + \epsilon$ is no longer a lower bound, i.e. there exist $x_0 \in S$* [4] *such that $x_0 < m + \epsilon$.*

   the greatest lower bound is denoted by $\inf S$ [5]*, called the infimum of S*
2. *A number M is called the least upper bound of S*
   - *$x \leq M$ for all x in S;*
   - *For any positive number ϵ, $M - \epsilon$ is no longer a upper bound, i.e. there exist $y_0 \in S$ such that $y_0 > M - \epsilon$.*

   the least upper bound is denoted by $\sup S$ [6]*, called the supremum of S.*

S: the logic sounds right to me. But how can you verify $m + \epsilon$ is not lower bound without knowing the value of ϵ?

T: We need some algebra rather than depending on numerical comparison. Take above $G(1/2)$ as an example, 0 is clearly a lower bound since each element $a_n = (1/2)^{n-1}$ in $G(1/2)$ is positive. To show 0 is the greatest lower bound, for any given $\epsilon > 0$, we can certainly take a sufficient large N to make $a_N = (1/2)^{N-1} < \epsilon$ so that $0 + \epsilon = \epsilon$ is no longer a lower bound [7]. Let us look at some other examples.

Example 1. 1. *$S = (1, 2)$. It is clear that* $\inf S = 1$ *and* $\sup S = 2$*.*
2. *$S = (1, 2, 3, ...)$ the set of all natural numbers. S is lower bounded with* $\inf S = 1$ *while it is not upper bounded and we write* $\sup S = \infty$

S: What makes $\sup S$ and $\inf S$ special?

[3]i.e. m is a lower bounder

[4]We follow the convention that x, y, z are used to denote ANY element in a set, while x_0, y_0, z_0 are used to denote a selected number.

[5]We write $\inf S = -\infty$ if it does not exist.

[6]We write $\sup S = \infty$ if it does not exist.

[7]For example, try to verify $N = max(2 - floor(log_2(\epsilon)), 1)$ works, where $floor(x)$ is the largest integer that is no larger than x

T: They are quite helpful for us to understand the fundamental concept **limit**. In fact, It turns out that they are exact the mysterious limit for a special type of sequences defined as follows.

Definition 1.3. *1. A sequence A in (1.1) is called increasing (decreasing) if $a_n \leq a_{n+1}$ ($a_n \geq a_{n+1}$) for any $n \geq 1$. A is called monotonic if it is either increasing or decreasing.*

2. The limit of a bounded increasing sequence A is defined to be $\sup A$, *denoted by*

$$\lim_{n\to\infty} a_n := \sup A; \tag{1.4}$$

3. The limit of a bounded decreasing sequence A is defined to be $\inf A$, *denoted by*

$$\lim_{n\to\infty} a_n := \inf A; \tag{1.5}$$

As you can see, for a monotonic sequence, the concept of limit is clear. One can visualize that " items in A approaches arbitrarily closely to the limit".

S: The phrase like "arbitrarily close" sound ambiguous to me.

T: You are not alone. Mathematicians in early days also got confused with such description. To clarify the ambiguity, let $O(l, \epsilon) = (l - \epsilon, l + \epsilon)$ denote the neighborhood of l with radius $\epsilon > 0$, which contains all numbers whose distances to l are less than ϵ as shown in Figure 1.

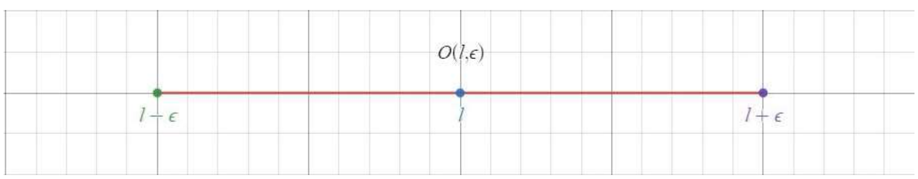

FIG 1. *The neighborhood $O(l, \epsilon)$ contains all numbers whose distances to l are less than ϵ*

Let $l = \inf A$ for a **bounded decreasing** sequence A. For any prescribed $\epsilon > 0$, since l is the greatest lower bound, $l + \epsilon$ is no longer a lower bound, which implies that there exists a_N in A such that $a_N < l + \epsilon$. Note that $l \leq a_N$ since l is lower bound, and therefore a_N is in $(l, l + \epsilon) \subset O(l, \epsilon)$. Since the sequence is decreasing, all terms after a_N is more closer to l than a_N and fall in the $(l, l + \epsilon)$ as shown in Figure 2, i.e.

$$|a_n - l| < \epsilon, \qquad \text{for ALL } n \geq N \tag{1.6}$$

S: Got it. items in A approach **arbitrarily close** to l simply because ϵ can be a arbitrarily small.

T: Exactly. let me emphasize that ϵ caps the distances to l for **all** terms following a_N. Pay attention to the key word "all". In another word, the distance threshold ϵ is applied to all items in the sequence except some *finite* many items in front of the sequence.

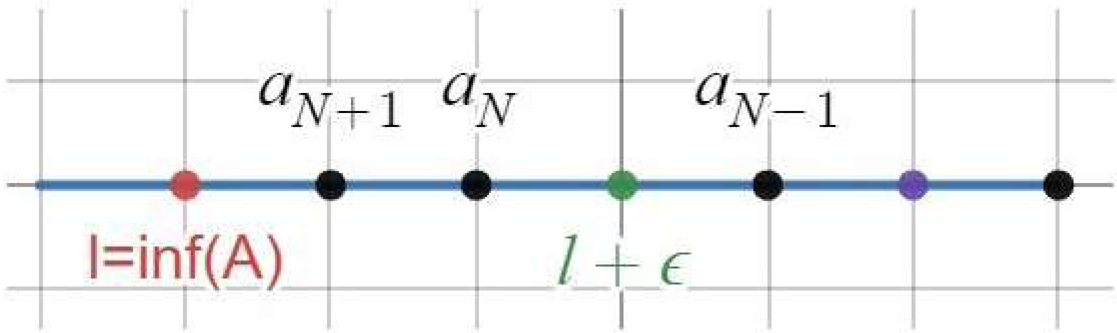

FIG 2. *The limit of a bounded decreasing sequence A is equal to $l = \inf A$. The quality ϵ is labeled as e in the graph.*

S: Let me confirm my understand. By saying "items in A approach arbitrarily close to l", it doesn't mean that certain **fixed** terms in the sequence are arbitrarily close to l. It actually means that, for any given threshold $\epsilon > 0$, all items except some finite many are close to l within the threshold ϵ. " arbitrarily close" is due to the fact that ϵ can be any small number without any restriction.

T: Exactly. It is equivalently to say that, for any $\epsilon > 0$, there exists an item a_N such that all following items $\{a_n; n \geq N\}$ are close to l within the threshold ϵ, i.e. $|a_n - l| < \epsilon, \forall n \geq N$. The idea can be used to describe the limit of a general sequence as follows

Definition 1.4. *A sequence A in 1.1 is called to* **converge** *to l if for any given $\epsilon > 0$, there exists N such that*

$$|a_n - l| < \epsilon, \qquad \forall n \geq N. \tag{1.7}$$

In this case, we write

$$\lim_{n\to\infty} a_n = l \qquad or \qquad a_n \to l \tag{1.8}$$

and l is called the **limit** *of the sequence.*

Remark 2. *The concept of the limit of a sequence is one of two fundamental concepts in calculus! We shall touch the other kind limit about function later.*

Let us look some examples.

Example 2.
1. $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. *It is increasing with* $\sup A = 1$
2. $\lim_{n\to\infty} \frac{1}{n^{2+(-1)^n}} = 0$. *It is neither increasing nor decreasing. For any $\epsilon > 0$, we need find required a_N. Since $0 < a_n \leq \frac{1}{n}$, to make $|a_n - 0| < \epsilon$,*

we only need $\frac{1}{n} < \epsilon$. If we choose N be any integer that is larger than $\frac{1}{\epsilon}$, then

$$|a_n - l| = a_n \leq \frac{1}{n} < \frac{1}{N} < \epsilon$$

for all $n \geq N$.

3. $\lim_{n\to\infty}(1 + (-1)^n)$ *does not exist. Intuitively, the sequence does not approach to a given value since a_n is alternating between 0 and 2. one can also argue vigorously that the sequence can not converge to any given number l. In fact, for any l, we have*

$$2 = |a_n - a_{n+1}| = |(a_n - l) + (l - a_{n+1})| \leq |a_n - l| + |a_{n+1} - l|$$

as such, $|a_n - l|$ and $|a_{n+1} - l|$ can not be smaller than $\frac{1}{2}$ simultaneously for any n. So for $\epsilon = \frac{1}{2}$, there is no N that meets the condition (1.7).

S: What can we do with limit? how does it bridge the finite world to the infinite world?

T: Let us start with the task to add all terms in A by Definition (1.1). It becomes actually quite trivial once we have the limit as a tool.

Definition 1.5. *Let $S_n = \sum_{1\leq i\leq n} a_i$ be the sum of the first n terms in A. If S_n converges to certain value S, then the sequence is called summable and S is defined as the sum of the sequence A. We write*

$$\sum_{1\leq i<\infty} a_i := S = \lim_{n\to\infty} S_n \tag{1.9}$$

Basically, we use the partial sum S_n as an estimation and the limit of the estimation process is equal to the exact value of the summation if the limit does exit [8]. As an example, consider a general geometric sequence

$$G(x) := (1, x, x^2, \ldots, x^{n-1}, \ldots). \tag{1.10}$$

we can add first n terms and get the partial sum

$$S_n := (1 + x + x^2 + \cdots + x^{n-1}) = \frac{1 - x^n}{1 - x}.$$

For $0 \leq x < 1$, it is not hard to show S_n is increasing and converges to its supreme $\frac{1}{1-x}$ [9], i.e.

$$\sum_{1\leq n<\infty} x^{n-1} = \frac{1}{1-x}, \quad 0 \leq x < 1 \tag{1.11}$$

S: To add infinite many items, we start working in the finite world by adding finite terms to get S_n, then take the limit of the sequence $(S_1, S_2, \cdots, S_n, \cdots)$ to get the sum of the sequence. Looks that limit indeed plays a bridge rule here.

T: Indeed! In general,

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[8] It is quite possible that the limit does not exist and hence the sequence is not summable. For example, let $a_n = (-1)^n$ and S_n is switching between 0 and -1, and hence does not converge.

[9] by proving $\lim_{n\to\infty} x^n = 0$ for $0 \leq x < 1$.

Statement 1.3. *If a direct solution for a task is not available, we often start with certain estimation process to estimate the target value and expect that the limit of the estimation process is the target value.*

S: But why bother to add infinite many terms?

T: It is one of basic operation for us to explore in the infinite world and is essential for both theoretic development and real world applications. But let us roam in the ancient Greek imaginary world for a while and think about the following famous paradox by the Greek philosopher Zeno nearly 2500 thousand years ago [10].

Discussion Question 1.1. Zeno's Paradox *Achilles (A), the fleet-footed hero of the Trojan War, is engaged in a race with a tortoise (T), which has been granted a head start. He shall never capture the tortoise since each time he reaches to the previous position of the tortoise, the tortoise has moved forward to a new position ahead of him as shown in Figure 3, where Achilles and the tortoise start at two positions A_0 and T_0 respectively and the tortoise moves ahead to T_i when Achilles catches up to $A_i = T_{i-1}$ for each $i = 1, 2, 3, ...$*

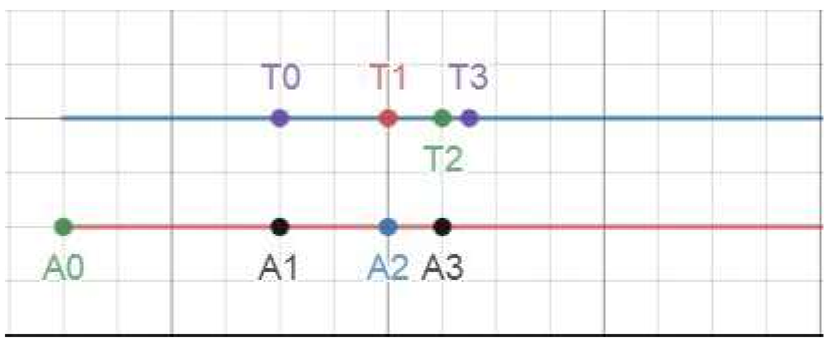

FIG 3. *Zeno's Paradox: The race between Achilles and a slow tortoise*

S: It is absurd and yet sounds logical.

T: It is a paradox, isn't it? You have to admire ancient Greeks for their passions and curiosities in seeking the knowledge. To get some insights, let us be more quantitative and assume that A runs and T moves at speed $v_A = 10m/s$ and $v_T = 1m/s$ [11] respectively and T has a head start of $100m$. Now assume that A spends t_i seconds to move from A_i to A_{i+1} for $i = 0, 1, 2, 3, ...$ It is clear

$$t_0 = \frac{100}{v_A} = 10$$

During the period $[0, t_0]$, T moves ahead by the distance $s_1 = t_0 \times v_T$. To cover the distance s_1, A needs $t_1 = \frac{s_1}{v_A} = t_0(\frac{v_T}{v_A})$ seconds to reach A_1. But in this period (t_1 seconds), T further moves forward by $t_1 \times v_T$ meters. In the next

[10]See Zeno's book "Achilles and the Tortoise"

[11]Let us assume that the tortoise moves really fast to make calculation easy.

step, A needs $t_2 = \frac{t_1 \times v_T}{v_A} = t_0(\frac{v_T}{v_A})^2$ seconds to reach the position A_2. In general, A needs $s_i = t_0(\frac{v_T}{v_A})^i$ seconds to reach A_i from A_{i-1}.

S: I see your points. But I rather directly calculate the total time that A needs to catch up the tortoise by simple algebra. Say A catches T after t seconds . Since A moves extra 100 meters than T, we have the equation

$$t \times v_A = t \times v_T + 100$$

and solve it for t

$$t = 100/(v_A - v_T) = 100/9.$$

T: Excellent. We know now that A should catch T in exactly 100/9 second. But remember that we have paradox to address. Do you have a way to find the total time following Zeno's argument?

S: By adding $t_1, t_2, ...$?

T: Right. We need add all those t_i to get the total time. Zeno's argument implicitly implies that adding infinite many of numbers should **always** lead to some sort of infinity. We now know how to add a geometric sequence in Equation 1.11 and find the total time T

$$T = \lim_{n\to\infty} T_n = \lim_{n\to\infty} 10(1 + \frac{v_T}{v_A} + \cdots + (\frac{v_T}{v_A})^{n-1}) = 10 \times \frac{1}{1 - \frac{1}{10}} = \frac{100}{9},$$

which is consistent to what you covered using simple algebra.

S: Woo! The phase "never capture" in the paradox is misleading because the total time A spends in the process is not infinity although Achilles's catch-up process goes on forever.

T: Right. mathematically, we can add infinite many of numbers and end up with a finite value if the sequence is summable as demonstrated here.

Now let us back to the real world. For a computer to handle arithmetic operations, numbers need to be expressed in decimal expression (or similar like binary format). For all irrational numbers, simple as $\sqrt{2}$, there are no pattern in their decimal expression. To approximate them with desired accuracy, we like to express a target number as the summation of a sequence of numbers so computer can effectively carry out the estimation. As a typical example, the well-known π can be expressed in following ways [12]

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}. \tag{1.12}$$

If we use the sum S_n of the first n term as approximation of $\pi/4$, one can show

[12] First discovered by Leibniz

[13]
$$|\frac{\pi}{4} - S_n| < \frac{1}{2n+1}$$
As such, we can not only estimate π by $4S_n$, but also know that the error is not bigger than $\frac{4}{2n+1}$. If you want to estimate π with tolerate error 10^{-3}, we need $\frac{4}{2n+1} < \frac{1}{10^3}$, or roughly $n > 2 \times 10^3 = 2000$. You are encouraged to search the latest development online to find some amazing algorithm to estimate π. The idea is the same: express π in term of a summation of series and use the partial summation of first n terms as approximation.

S: So we end with an irrational number by adding infinite many rational numbers. I remember that we can only get a rational number by adding finite many rational numbers.

T: The expression (1.12) of π shows that you might get an apple by adding infinite many of oranges. But what might surprise you more occurs when we try to express sophisticated functions in the summation of simple power functions as we did in Equation (1.11) where the function $\frac{1}{1-x}$ $(x \in (0, 1))$ is expressed as a summation of infinite many of power functions. We are going to show in Section 7 that, under certain conditions, this can be done for a general function $f(x)$, i.e. there exists $a_0, a_1, ..$ such that

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \tag{1.13}$$

For examples [14],

$$\sin(x) = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \dots, \quad x \in R \tag{1.14}$$
$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \dots, \quad x \in R \tag{1.15}$$

S: Interesting. I do not know much about the two trigonometric functions except they are periodic and bounded by 1. It is hard for me to associate them with power functions.

T: It is actually quite easy to derive Eq (1.14) and (1.15) once we develop certain tools. We will see more magics in our future exploration in the infinite world. We have made the first step in the journey although we still have a long

[13]
$$\begin{aligned}
|\frac{\pi}{4} - S_n| &= |\frac{1}{2n+1} - \frac{1}{2n+3} + \frac{1}{2n+5} - \frac{1}{2n+7} + \dots| \\
&= |(\frac{1}{2n+1} - \frac{1}{2n+3}) + (\frac{1}{2n+5} - \frac{1}{2n+7}) - \dots| \\
&= (\frac{1}{2n+1} - \frac{1}{2n+3}) + (\frac{1}{2n+5} - \frac{1}{2n+7}) - \dots \\
&= \frac{1}{2n+1} - (\frac{1}{2n+3} - \frac{1}{2n+5}) - (\frac{1}{2n+7} - \frac{1}{2n+9}) - \dots \\
&< \frac{1}{2n+1}
\end{aligned}$$

[14] $\sin(x)$ or $\cos(x)$ will be defined and studied in later chapters.

way to go. Let me finish this section with what need to be done in future. We need

Task 1.1.

1. *develop the tools to justify whether a sequence converges rather than just depending on checking by definition;*
2. *develop the tools to find the limit value of a general sequence if it exists;*
3. *estimate the limit of a sequence if there is no chance to find the exact value and handle the estimation error effectively.*
4. *know how to represent a sophisticated function by simple functions as in 1.13 so that we can handle them effectively if we can deal with those building blocks of functions.*