Preface

0.1. To whom this dialogue is written for?

This dialogue is intended for a beginner in calculus who

- is not satisfied with incomplete stories or "sound right" overview on calculus, but takes this subject seriously to develop a solid foundation for their further studies in science and technologies;
- is familiar with simple functions like $y = x^2$ and comfortable with basic algebra;

The development of the foundation of calculus built in this dialogue spans more than 150 years and is contributed by quite a few great mathematicians. It will almost surely take time for beginners to be able walking comfortably in a totally different world (infinite world) from the world (finite world) the journey begins. The patience to read through the dialogue is essential to accomplish the journey. As such, in addition to above two requirements,

- can tolerate a 50-page book and to read it carefully if the reader is not genius and has no much experience on calculus.

0.2. What does the author aim to achieve?

The author aims to provide readers a dialogue on calculus with following features:

1. It covers comprehensively key concepts, fundamental results, and analytical skills in calculus such that readers don't need turn to other references to understand what is covered in the dialogue and can dependent on themselves for their further study of this subject in the future;
2. Key ideas and fundamental results are demonstrated by intuitive samples with thorough discussion to address subtle issues for solid understanding;
3. All covered concepts are clearly defined without ambiguity and theorems are proved vigorously to help the readers get familiar with the analytical skills that are commonly used in calculus.

In a nutshell, calculus is all about limit and a good understanding of limit is half-way to master the subject. Technical part is likely no longer a hurdle once we fully understand the concept and get familiar with a few tricks because the same idea is used almost everywhere in calculus:

Statement 0.1. *For the most of the problems in calculus world, we usually do not have a DIRECT way to find the EXACT solution, denoted by S, of the problem. Hence, we start with certain estimation* **process**, *each step of the process is done in such way that it tends to provide somehow better approximation to S than the estimations in previous steps. The estimation process is managed to be endless such that we can always improve the existing estimations to obtain next*

better estimation. It is expected that the estimation process will generally lead to the target value S under certain conditions.

In fact, the limit theory is applied to effectively obtain S from the estimation process. Once you become comfortable with limit, you play similar tricks to all problems and eventually have the control in applying calculus as a powerful tool to solve problems. As such, calculus is supposed to be relatively easy.

In reality, calculus is generally considered as a hard subject for many beginners in high school and among college students. What makes calculus a challenging subject, if not a hard one?

Development history of calculus might be helpful for us to find the answer. In their initial works on calculus in the seventeen century, neither Newton nor Leibniz, the two founders of calculus, was clear in their definitions of the key concepts (derivative and differential). Newton's approach to the calculus is what he described as the method of infinitesimals, a method that is "shortly explained rather than accurately demonstrated" as he acknowledged ([1]). Leibniz depended on so called infinitely small quantities, which were characterized as "not zero, but were smaller than any finite quantity" ([1]). Certain ambiguous phrase such as "approach indefinitely", "as little as one wishes", and "last ratios of infinitely small increments" were used in the description of certain key concept, which made the subject hard to understand even within mathematicians through eighteen century ([1]).

The lesson is that ambiguities on the key concept of limit cause the confusion and make the subject hard even for those pioneers. In fact, once the concept of limit was clarified by German mathematician Weierstrass about 150 year later after the Newton and Leibniz initial works, the foundation can be built relatively easily as shown in this dialogue. Calculus turns out to be a perfect example to show how much a great idea can influence the development of mathematics.

Helping readers to understand the idea of limit is the first goal of this dialogue. In introducing each of important concepts, we start the dialogue with some demonstrative examples to help readers visualize the idea. But we are not satisfied with "look right", rather aim for "logically correct". The foundation of calculus should not be on geometry, but on algebras as Gauss, one of the greatest mathematicians, suggested ([1]). A quite amount of discussion in the dialogue aims to clarify certain subtle concepts that are essential to avoid ambiguity. The discussion, even at the cost of extra precious pages, is worth for the beginners and is where we ask for the extra patience from the readers.

Once readers are comfortable of the key concepts, they should be ready to enjoy the beauty of the theory and appreciate how powerful tool calculus can be by one magic after another.