7. High Order Derivatives of Functions and Taylor Theorem

T: For a given function $f(x)$, as discussed before, the derivative function $f'(x)$ contains the information whether the function increases or decreases over certain range and where it reaches to maximum or minimum values at certain point (see Figure8). That information is important, but often not sufficient. For example, a function can increase in two different ways associated with either decrease of slope or increase of slope as shown in Figure 18. Notice that the change of the slope can be measured by the derivative of the slope, i.e. $(f'(x))'$, which is defined as the second derivative and denoted by $f^{(2)}(x)$ or $f''(x)$.

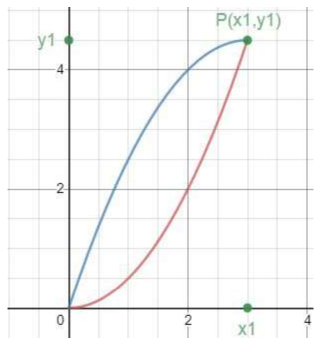

FIG 18. *The first derivative is positive for both curves that are increasing. The second derivative for the red curve is positive and reflect the concave upward character. The blue curve has negative second derivative and is concave downward.*

S: So there are some other things to look at in addition to the monotonic property of the function.

T: Right. The sign of the first derivative decides whether it increase or decrease, the sign of the second derivative decides so called convexity of the function. The function is called concave upward or concave downward depending if $f^{(2)} > 0$ or $f^{(2)} < 0$. A typical concave-upward function is $f(x) = x^2$ with $f'(x) = 2x$ and $f''(x) = 2$ while $f(x) = -x^2$ represents a typical concave downward function with $f''(x) = -2$.

S: Should we also think about other derivatives like third derivative or fourth derivative?

T: We do need other order derivatives as shown in an important application below.

Definition 7.1. *we define inductively n-th derivative $f^{(n)}$ of f, if existing, as follows*

$$f^{(n)} = (f^{(n-1)})'(x), \quad n = 1, 2, \ldots \tag{7.1}$$

where $f^{(0)} = f(x)$ and $f^{(1)} = f'(x)$.

Example 8. *if* $f(x) = (x - x_0)^n$, *then*

$$\begin{aligned} f'(x) &= n(x-x_0)^{n-1} \\ f^{(2)} &= n(n-1)(x-x_0)^{(n-2)} \\ f^{(k)} &= n(n-1)\cdots(n-k+1)(x-x_0)^{n-k}, \quad 1 \le k \le n \end{aligned} \tag{7.2}$$

Proposition 7.2. *If* $f(x)$ *has up to* $n+1$*-th derivatives over* $[a, b]$ *and* $x_0 \in (a, b)$ *such that*

$$f'(x_0) = f^{(2)}(x_0) = \cdots = f^{(n)}(x_0) = 0,$$

then for $x \neq x_0$ *there exists* c *between* x_0 *and* x *such that*

$$\frac{f(x)}{(x-x_0)^{(n+1)}} = \frac{f^{(n+1)}(c)}{(n+1)!} \tag{7.3}$$

Proof.

$$\begin{aligned} \frac{f(x)}{(x-x_0)^{(n+1)}} &= \frac{f(x) - f(x_0)}{(x-x_0)^{(n+1)}} \\ &= \frac{f'(x_1)}{(n+1)(x_1-x_0)^n}, \quad x_1 \in (x_0, x) \text{ by Theorem 2.4 } ^{31} \\ &= \frac{f'(x_1) - f'(x_0)}{(n+1)(x_1-x_0)^n} \\ &= \frac{f''(x_2)}{(n+1)n(x_2-x_0)^{(n-1)}}, \quad x_2 \in (x_0, x_1) \text{ by Thm 2.4} \\ &\vdots \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}, \quad c \in (x_0, x_n) \text{ bu using Thm 2.4 } n+1 \text{ times} \end{aligned}$$

□

Theorem 7.3. Taylor Formula *If* $f(x)$ *has up to* $n+1$ *derivatives exist over certain neighborhood* $O(x_0, r) = (x_0 - r, x_0 + r)$ *of* x_0*, then for any* $x \in O(x_0, r)$*, there exists* $c \in O(x_0, |x - x_0|)$ *such that*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n \tag{7.4}$$

where

$$R_n(x) := \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{(n+1)}. \tag{7.5}$$

Proof. Let

$$F(x) := f(x) - f(x_0) - f'(x_0)(x-x_0) - \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 - \cdots - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

One can show by Eq. 7.1

$$F(x_0) = F'(x_0) = F^{(2)}(x_0) = \cdots = F^{(n)}(x_0) = 0$$

and $F^{(n+1)}(x) = f^{(n+1)}(x)$ for all x, which implies Eq (7.4) by Proposition 7.2. □

S: By Taylor formula, a general function can be approximated by a n degree polynomial with residual R_n.

T: Right! Taylor formula 7.4 is important in computational science. If $|f^{(n+1)}(x)|$ can be bounded by some constant M, we can estimate $f(x)$ by a n order polynomial

$$S_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

The estimation error is smaller than

$$\frac{M}{(n+1)!}(x - x_0)^{(n+1)}.$$

Notice that **the polynomial $S_n(x)$ only involve arithmetic operations which can be carried out by a computer.** As a special case, we have

Corollary 7.4. *If $f(x)$ is smooth on $[a, b]$ in the sense that $f^{(n)}(x)$ exists for any n and $|f^{(n)}(x)| \leq M$ for any $x \in [a, b]$ and n, then*

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{7.6}$$

where x_0 is a fixed point in (a, b).

Proof. For any fixed $x \in [a, b]$, we need to show that

$$\lim_{n \to \infty} S_n(x) = f(x)$$

but

$$|S_n(x) - f(x)| = |R_n(x)| < \frac{M}{(n+1)!}|(x - x_0)^{(n+1)}| < \frac{M}{(n+1)!}(b - a)^{n+1}$$

For any $\epsilon > 0$, it is not hard to see that there exists N such $\frac{M}{(n+1)!}(b-a)^{n+1} < \epsilon$ for all $n > N$. □

Finally, for two basic trigonometric functions $\sin(x)$ and $\cos(x)$, it can be shown that

$$\sin'(x) = \cos(x), \quad \cos'(x) = -\sin(x),$$

which implies

$$\sin^{(2n+1)}(0) = (-1)^n, \quad \sin^{(2n)}(0) = 0, \quad n = 0, 1, 2, \ldots$$

and

$$\cos{(2n+1)}(0) = 0, \quad \cos^{(2n)}(0) = (-1)^n, \quad n = 0, 1, 2, \ldots.$$

The equations (1.14) and (1.15) is then directly from the formula (7.6).

So far, We have explored in the infinite world for a quite while and kind of reach to a milestone for essential concepts and fundamental results.

T: WOW!!! I can not believe that I can go so far with you. What a great job to push limit on me! I have to admit that I need some time to think through and absorb all of these. But many thanks for arousing my interests in this subject.

S: Good to hear that and see you next time!

References

[1] Morris Kline, Mathematical Thought From Ancient to Modern Times, Oxford University Press, 1972